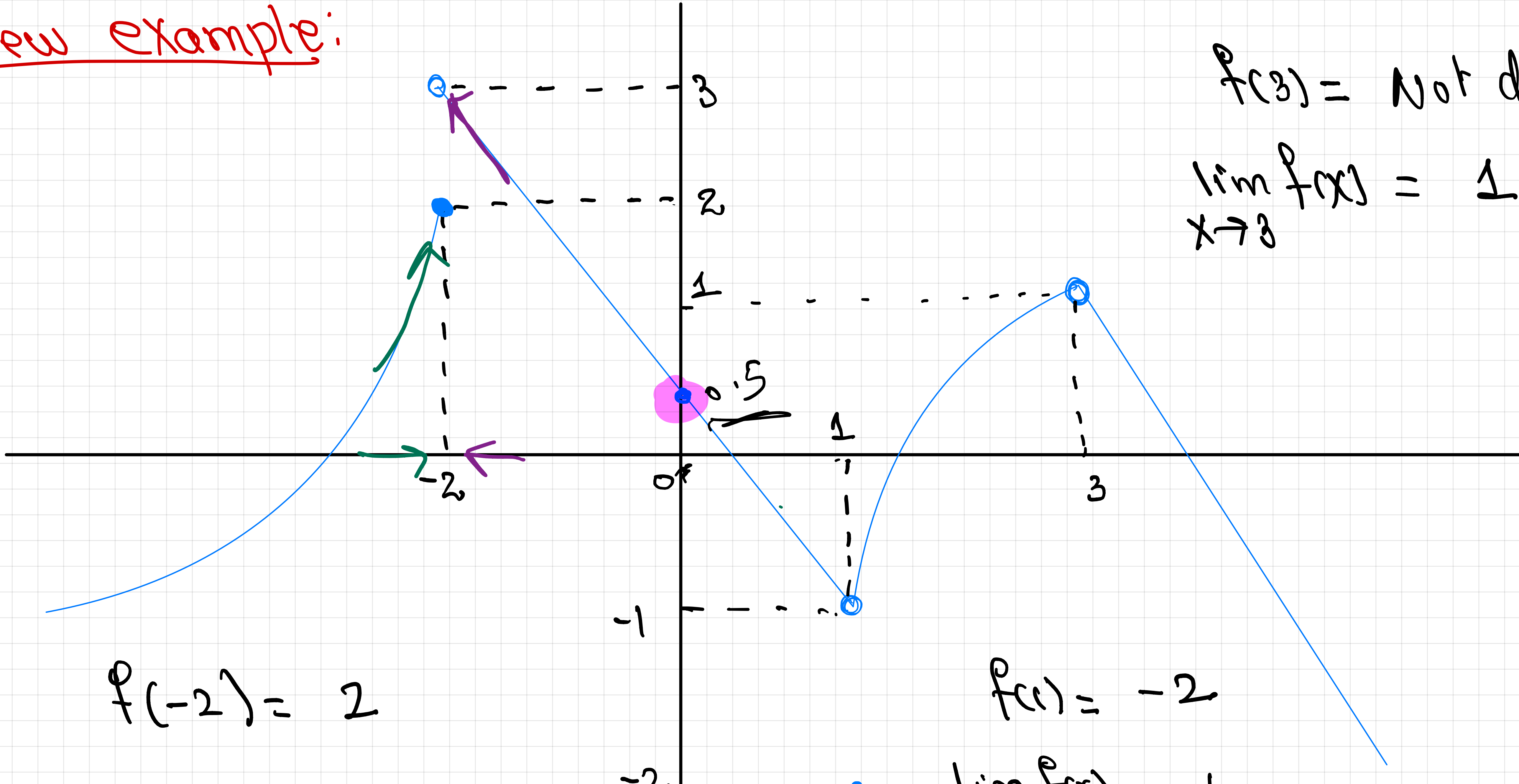
MA 161 - Lesson 5 (2.3)

Today: Computing Limits
Limit Laws
Squeeze/Sandwich Theorem

Office Hours: Monday, Wednesday, Friday 2:45pm - 4:15pm

Announcements: Exam 1 on Thursday 09/18, 8pm - 9pm
HW5 Due: Tuesday 09/09
Quiz 3 (Lesson 4): Tuesday 09/09

Review example:



$f(3) =$ Not defined

$\lim_{x \to 3} f(x) = 1$

$f(-2) = 2$

$\lim_{x \to -2} f(x) = \underbrace{DNE}$

Left & Right limit are Not Same

$f(1) = -2$

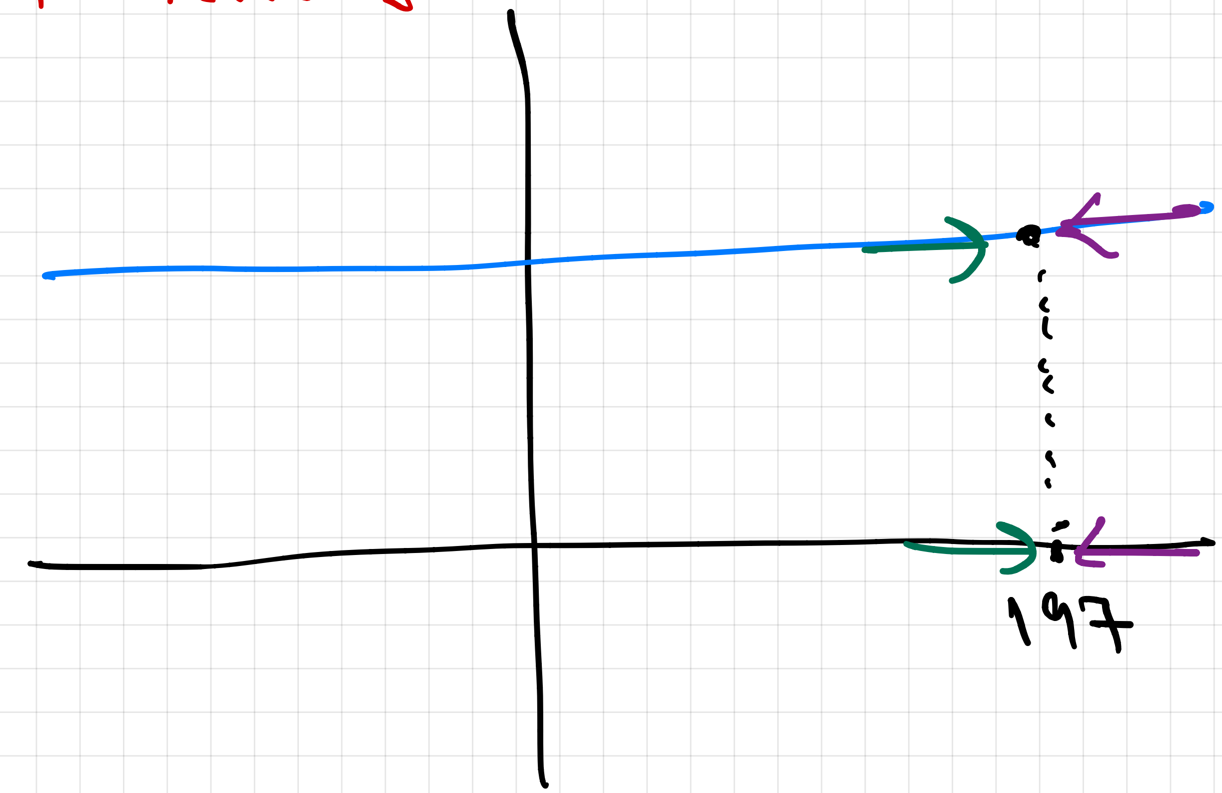
$\lim_{x \to 1} f(x) = -1$

$f(0) = 0.5 = \lim_{x \to 0} f(x)$

Finding limits of simple functions

eg: $f(x) = 7 \rightsquigarrow$ constant

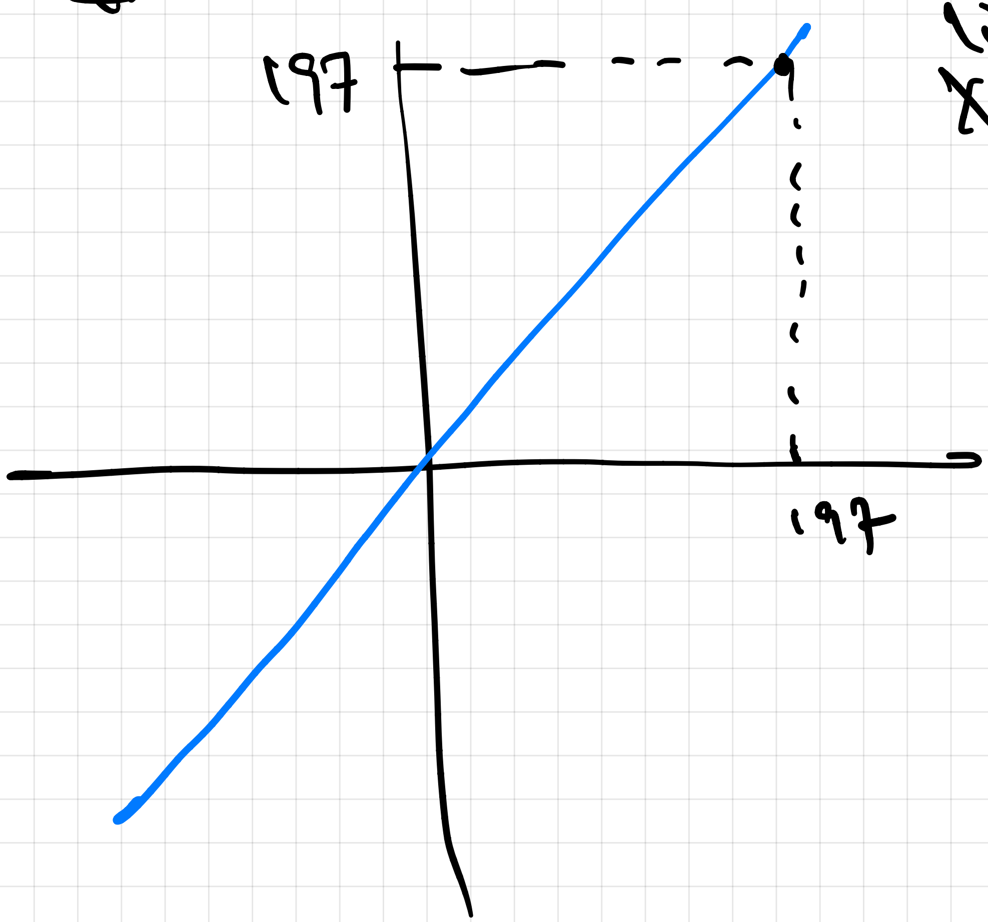
$$\lim_{x\to 197} f(x) = \lim_{x\to 197} 7 = 7$$



Suppose $f(x) = c$ a constant

$$\lim_{x\to a} c = c.$$

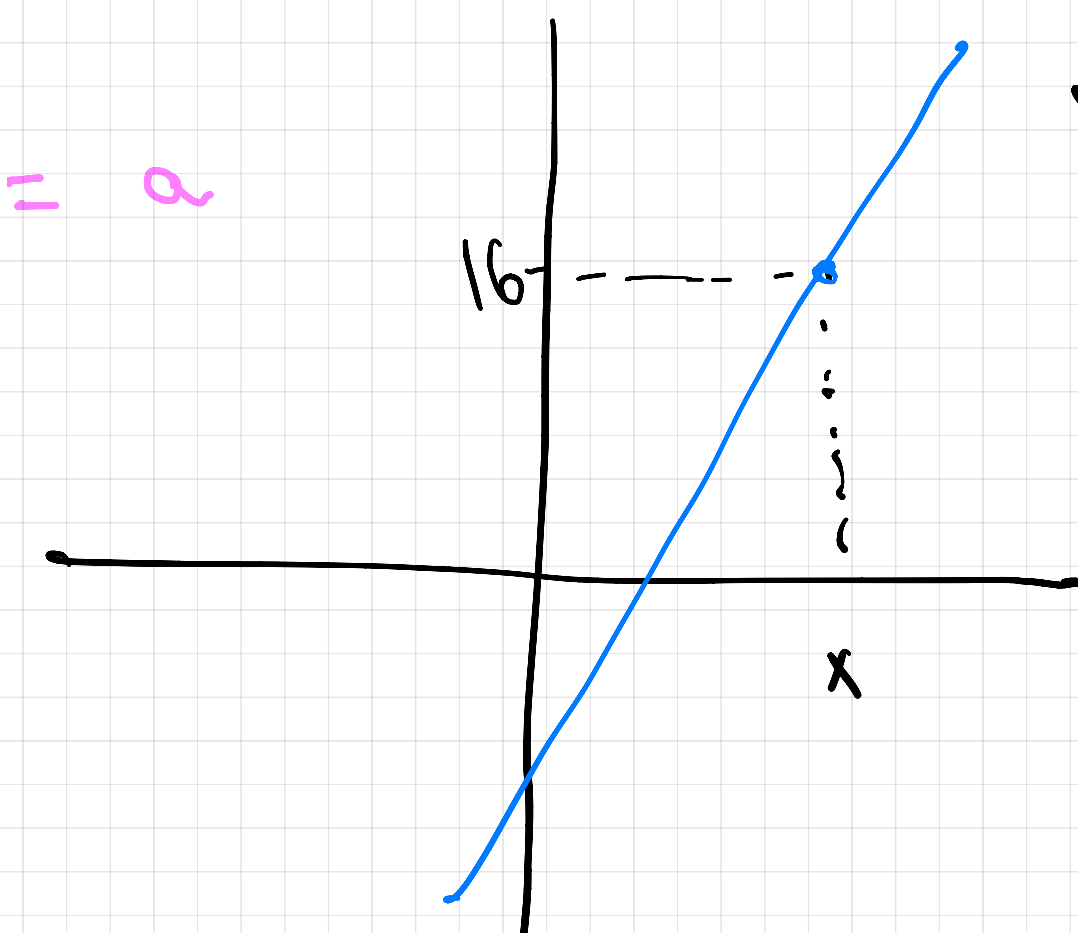
eg: $f(x) = x$



$$\lim_{x\to 197} x = 197$$

$$\lim_{x\to a} x = a$$

eg: $f(x) = 7x - 5$



$$\lim_{x\to 3} 7x-5 = 7(3)-5$$
$$= 16$$

Let $f(x) = mx+b$

$$\lim_{x\to a} f(x) = ma+b.$$

Finding limits of polynomial functions

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

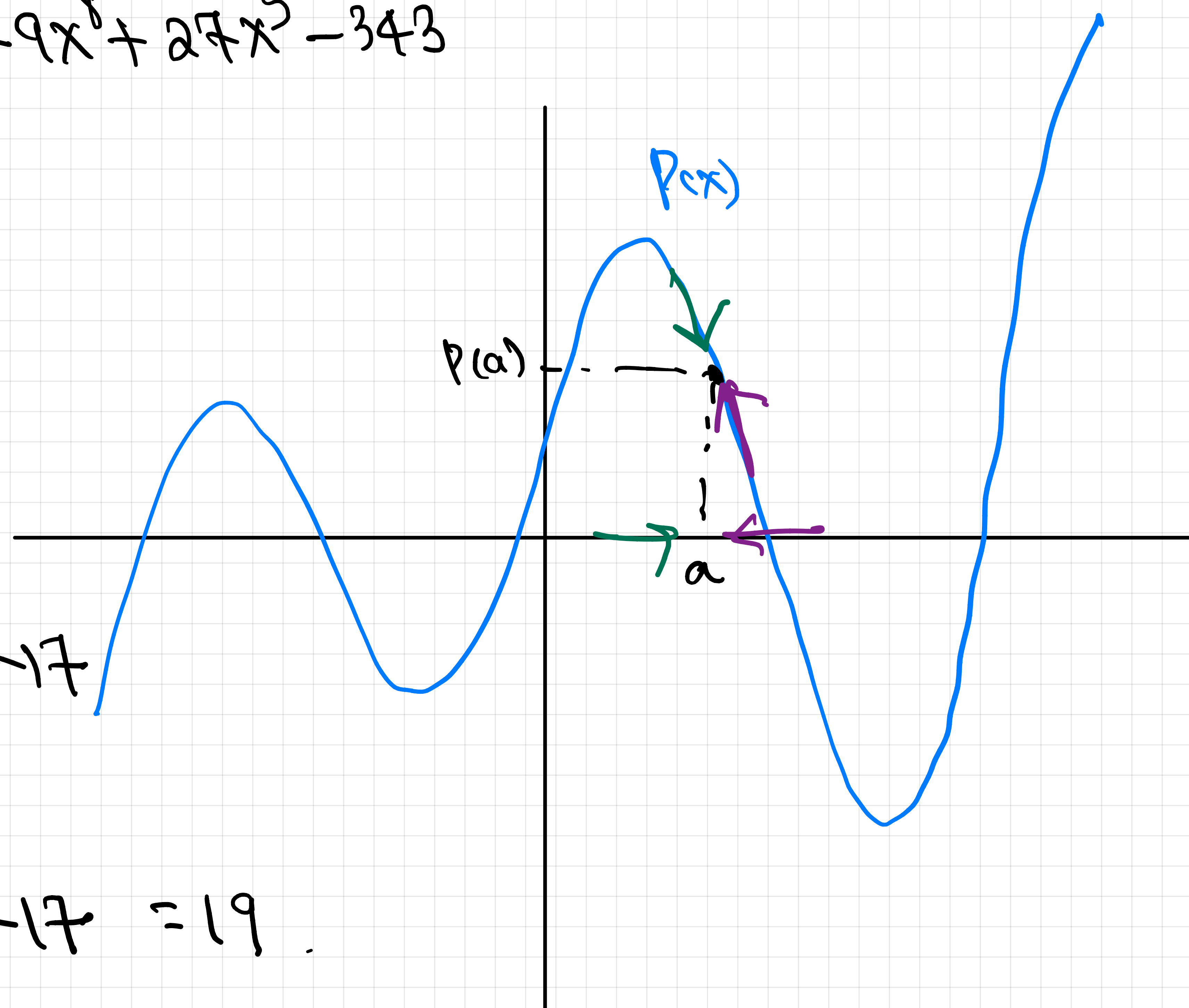
w/ degree n polynomial.

eg: $x^3 + 7x$

$x^{10} - 9x^8 + 27x^5 - 343$

Suppose $P(x)$ is a polynomial

$$\lim_{x \to a} P(x) = P(a).$$



eg: $P(x) = x^3 + 3x - 17$

$$\lim_{x \to 2} P(x) = P(2) = 2^3 + 3(2) - 17$$

$$= -3.$$

$$\lim_{x \to 3} P(x) = P(3) = 3^3 + 3(3) - 17 = 19.$$

$$\lim_{x \to 2} x = 2$$

$$\lim_{x \to 2} x^3 = \left(\lim_{x \to 2} x\right)^3 = 2^3 = 8$$

$$\lim_{x \to 2} 3x = \left(\lim_{x \to 2} 3\right) \cdot \left(\lim_{x \to 2} x\right) = 3 \times 2 = 6$$

$$\lim_{x \to 2} 17 = 17$$

$$p(x) = x^3 + 3x - 17$$

$$\lim_{x \to 2} (x^3 + 3x - 17) = \lim_{x \to 2} x^3 + \lim_{x \to 2} 3x - \lim_{x \to 2} 17$$

Limit Laws: Suppose $\lim_{x \to a} f(x) = L_1$, $\lim_{x \to a} g(x) = L_2$

① $\lim_{x \to a} \left(f(x) + g(x) \right) = L_1 + L_2$

② $\lim_{x \to a} \left(f(x) - g(x) \right) = L_1 - L_2$

③ $\lim_{x \to a} \left(f(x) \cdot g(x) \right) = L_1 \cdot L_2$

④ $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{L_1}{L_2}$, when $L_2 \neq 0$

⑤ $\lim_{x \to a} \left(f(x) \right)^K = L_1^k$

⑥ $\lim_{x \to a} \sqrt[K]{f(x)} = \sqrt[k]{L_1}$

Finding limits of Rational functions

$R(x) = \dfrac{P(x)}{Q(x)}$, $P(x)$ & $Q(x)$ are polynomials.

Suppose $Q(a) \neq 0$ ⇝ $x = a$ is in the domain of $R(x)$

$$\lim_{x \to a} P(x) = P(a) \quad , \quad \lim_{x \to a} Q(x) = Q(a)$$

$$\lim_{x \to a} R(x) = \lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} .$$

eg: #: $\displaystyle \lim_{x \to 8} \frac{x^2 - 4x - 21}{x - 7}$ $x = 8$ in the denominator $8 - 7 \neq 0$.

plug in $x = 8$ $= \dfrac{8^2 - 4(8) - 21}{8 - 7} = 11.$

e.g.: $\lim_{x \to 7} \frac{x^2-4x-21}{x-7}$ } plug in $x=7$ ⇝ $\frac{7^2-4(7)-21}{7-7} = \frac{0}{0}$

indeterminate form

$x^2-4x-21=0$ when $x=7$ ⇝ $x-7$ is a factor of $x^2-4x-21$

Factorize! $x^2-4x-21 = (x-7)(x+3)$

when $x \neq 7$ ⇝ $\frac{x^2-4x-21}{x-7} = \frac{\cancel{(x-7)}(x+3)}{\cancel{(x-7)}} = x+3$

$$\lim_{x \to 7} \frac{x^2-4x-21}{x-7} = \lim_{x \to 7} x+3 = 7+3 = 10$$

limit is function value close to $x=7$ NOT at $x=7$.

plug in because $x+3$ is a polynomial.

Rationalization

$$\sqrt{5}-\sqrt{3} = \frac{\boxed{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}}{(\sqrt{5}+\sqrt{3})}$$

$$(a-b)(a+b) = a^2-b^2$$

$$= \frac{5-3}{\sqrt{5}+\sqrt{3}} = \frac{2}{\sqrt{5}+\sqrt{3}}$$

eg:

$$\lim_{x\to 0} \frac{\sqrt{x+25}-5}{x}$$

Plug in $x=0$ $\frac{\sqrt{25}-5}{0} = \frac{0}{0}$

$x \neq 0$

$$\frac{\sqrt{x+25}-5}{x} = \frac{(\sqrt{x+25}-5)(\sqrt{x+25}+5)}{x(\sqrt{x+25}+5)} = \frac{x+\cancel{25}-\cancel{25}}{x(\sqrt{x+25}+5)}$$

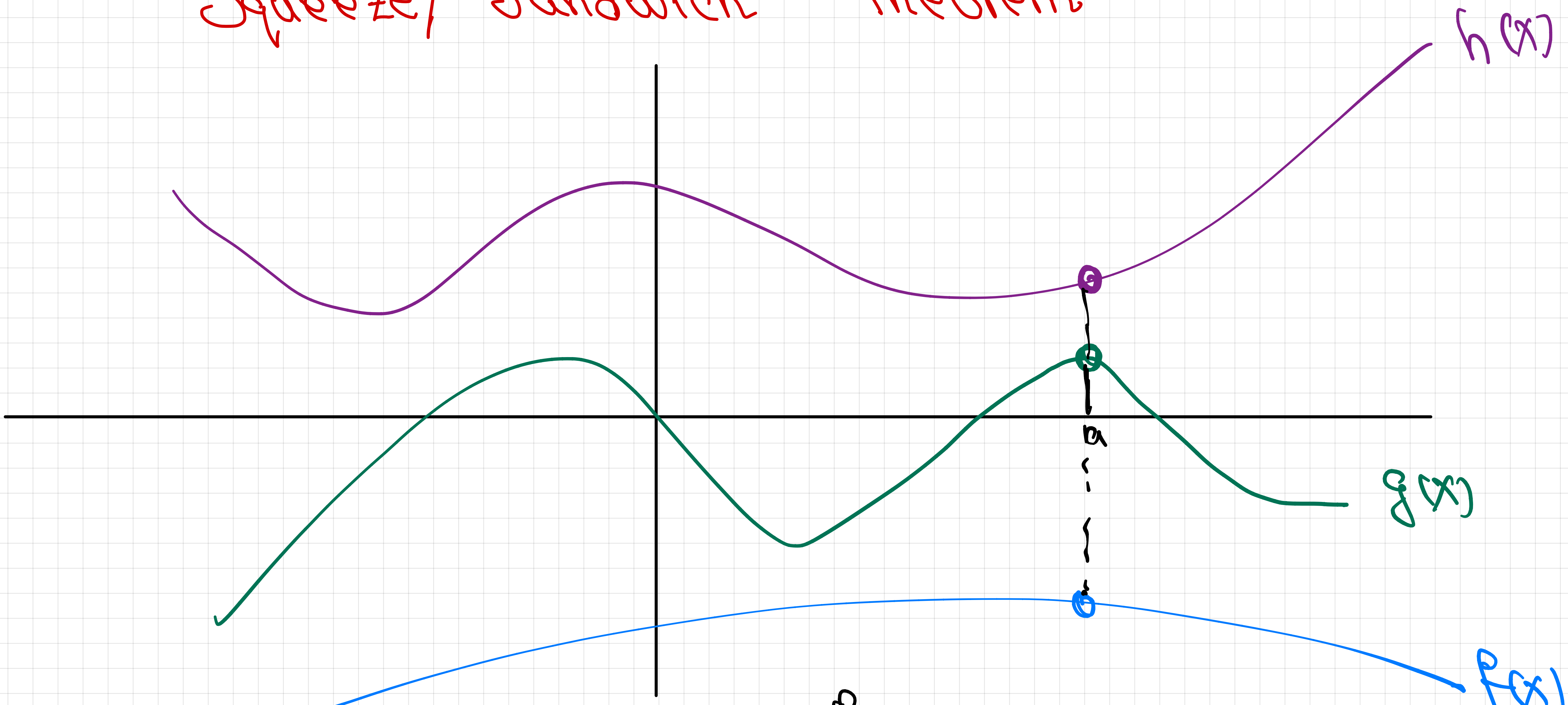
$$= \frac{\cancel{x}}{\cancel{x}(\sqrt{x+25}+5)}$$

$$= \frac{1}{\sqrt{x+25}+5}$$

$$\lim_{x\to 0} \frac{\sqrt{x+25}-5}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+25}+5} = \frac{1}{\sqrt{25}+5} = \frac{1}{10}$$

eg! Evaluate $\lim_{x \to 3} \dfrac{\frac{1}{x} - \frac{1}{3}}{x - 3}$

Squeeze/ Sandwich Theorem



Near $x = a$ Suppose $f(x) \le g(x) \le h(x)$

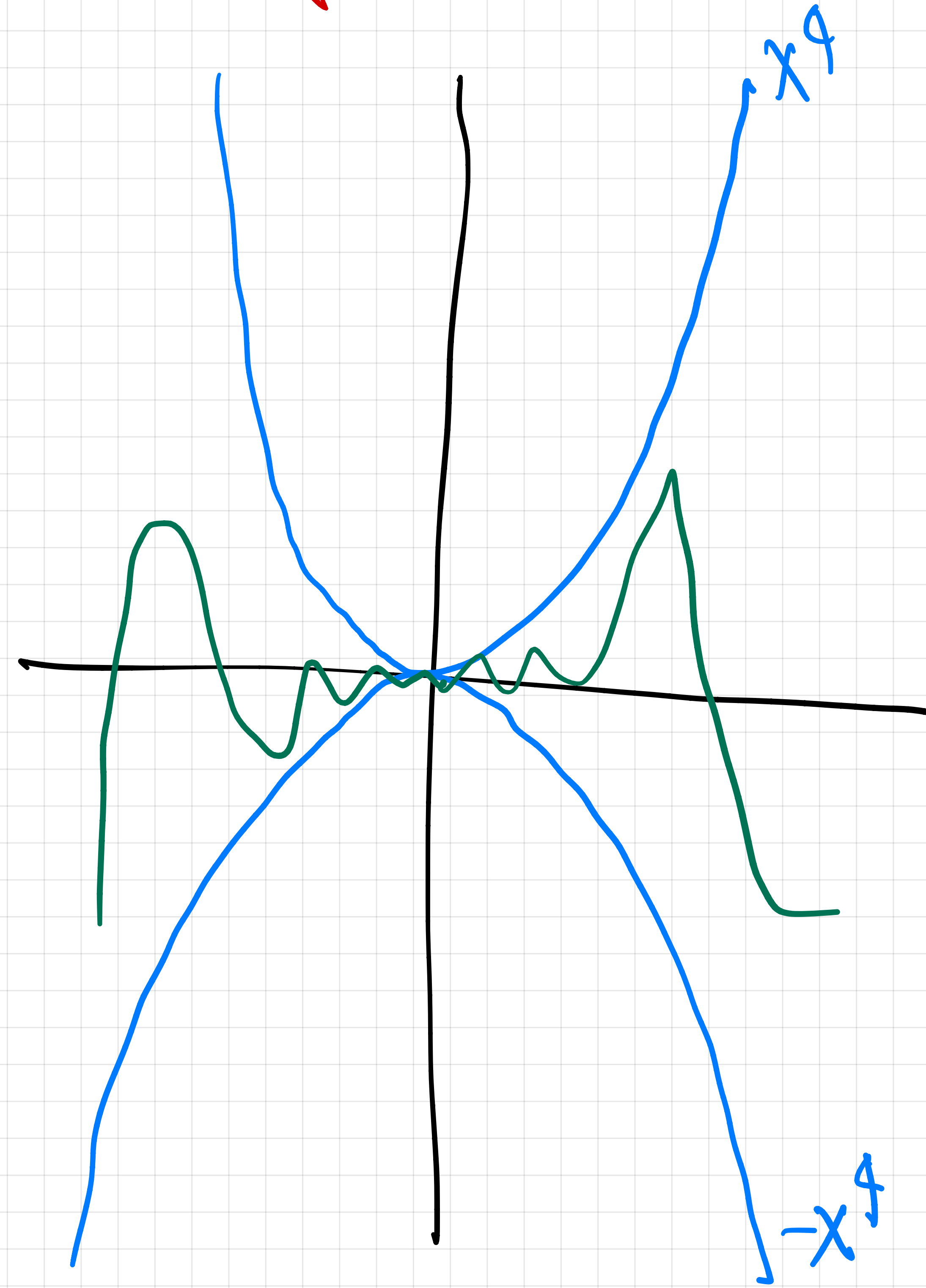
$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x) \le \lim_{x \to a} h(x)$$

e.g.: Evaluate $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right)$] CANNOT Plug in 0.

for any θ $-1 \le \cos\theta \le 1$

for any $x \neq 0$ $-1 \le \cos\left(\frac{2}{x}\right) \le 1$

Multiply with x^4 $-x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4$



$$\lim_{x \to 0} (-x^4) = 0 = \lim_{x \to 0} x^4$$

$\therefore$ By Squeeze theorem

$$\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0.$$